The gravitational cusp anomalous dimension from AdS space

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Abstract

Recently a new picture has been developed for examining Wilson lines, and the corresponding anomalous dimensions which govern their renormalization properties. By making a particular coordinate transform, the calculation of the cusp anomalous dimension in QED or QCD can be related to the energy of a pair of static charges in Euclidean Anti-de-Sitter (AdS) space. This paper shows how the same picture can be used to describe Wilson lines in quantum gravity. We show how the relevant cusp anomalous dimension (which has recently been shown to be one loop exact) can be obtained using the Newtonian limit of General Relativity. We also show how both the QED and gravity cases emerge as special cases of a general formulation, and that a continuous parameter exists which interpolates between them. The results may be useful in examining the relations between gauge and gravity theories.

1 Introduction

Wilson lines have been studied for many years in a variety of contexts [1–10], in both Abelian and non-Abelian gauge theories. In particular, they govern the structure of large logarithms in perturbation theory due to soft gluon emission [11, 12] as is also seen in other approaches such as soft collinear effective theory (SCET) [13–20], factorisation theorems [21–23], or the path integral technique of [24]. Essentially, hard partons emitting soft gluons cannot recoil and thus can only change by a phase. For this phase to have the right gauge transformation properties to slot into an amplitude, it must be a Wilson line evaluated along a contour given by the hard momentum of the outgoing particle. The ultraviolet renormalisation properties of Wilson lines govern the infrared singularities of scattering amplitudes [5–7, 25–28], a fact which has recently been used to derive an all-order ansatz for the infrared singularities of QCD scattering amplitudes [29–33], whose consequences have been discussed further in [34–37]. More exotic applications of Wilson lines come from the interface between gauge and string theory, where it has recently been conjectured that certain Wilson loop configurations are dual to scattering amplitudes in $\mathcal{N} = 4$ Super-Yang Mills theory at strong (and weak) coupling [38, 39].

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Correlators of Wilson line operators have ultraviolet (UV) divergences. For smooth contours, these are completely removed by renormalisation of the gauge theory coupling. For contours which contain cusps or intersections, however, additional UV singularities are present which, after renormalisation of the coupling, factor out into an overall multiplicative factor [3–5]. Wilson line correlators are then governed by renormalisation group equations, involving an anomalous dimension. For the case of a single cusp, the latter is known as the *cusp anomalous dimension*, which has become a quantity of central importance in the study of infrared singularities and resummation.

In Abelian theories, the all-order structure of the cusp anomalous dimension is dictated by a simple subset of possible Feynman diagrams, namely those in which the Wilson line contours meeting at the cusp are joined by a connected subgraph. The only possibilities are single photon emissions, or photon emissions which are joined by fermion loops. Traditionally this result was derived using the eikonal identity for multiple soft photon emissions [21]. Recently, an alternative proof has been given, using path integral techniques that relate the exponentiation of connected subgraphs in Wilson loops with a cusp, to the textbook exponentiation of connected diagrams in a general quantum field theory [24]. This simple structure is referred to as *Abelian exponentiation* in the literature, and in particular implies that if there are no propagating fermions, the cusp anomalous dimension is one-loop exact. Consequently, the infrared singularities of scattering amplitudes are purely governed by the exponentiation of the one-loop result. A similar result holds for correlators involving more than two Wilson lines meeting at a point. Then, the relevant anomalous dimension is completely determined to all orders by connected subdiagrams spanning the outgoing particles. Again, one-loop exactness follows if fermion loops are absent.

In non-Abelian theories, the cusp anomalous dimension is, unsurprisingly, more complicated. Nevertheless, its all order structure is still dictated by a subset of Feynman diagrams, which permit a simple topological classification: as well as connected subdiagrams spanning the outgoing Wilson line contours, one may also have diagrams which are *two-eikonal line irreducible*[3]. Such diagrams have modified colour factors which are maximally non-Abelian, and are referred to as *webs* in the literature [40–42][4]. Another way to think about webs is that they are, by definition, those diagrams which enter the exponent of the Wilson line correlator. Their structure is thus usually referred to as *non-Abelian exponentiation*, by analogy with the Abelian case. These results were recently reconsidered in [24], where an alternative derivation of webs was given using statistical physics methods. Furthermore, the notion of webs has recently been extended to correlators involving many Wilson lines meeting at a point, by two groups of authors [44, 45]. By analogy with the two line case (alternatively, a single contour with a cusp), webs are those diagrams which enter the exponent of the Wilson line correlator. However, their structure is markedly different to the two line case. The most significant difference is that *reducible* diagrams contribute to the exponent in general. Furthermore, webs become closed sets of diagrams whose members are related by gluon permutations. Their colour and kinematic information is entangled to all orders in perturbation theory by so-called *web mixing matrices* [45], whose structure is in principle governed purely from combinatorics [46]. This structure has yet to be fully explored, but is certainly crucial to understanding the all-order structure of infrared singularities in multileg scattering amplitudes [47].

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[3]That is, subdiagrams which cannot be disconnected by a single cut through both eikonal lines.

[4]A novel geometric interpretation of webs in two parton scattering has very recently been presented in [43].

Recently, a new way to think about Wilson lines was presented in [48]. The authors start off by considering a collection of Wilson lines meeting at a point in Minkowski space. They then transform to a set of *radial coordinates*, which map Minkowski space to $\mathbb{R} \times$ AdS, where AdS denotes Euclidean anti-de-Sitter space. Parametrisation invariance of the Wilson line contours in Minkowski space becomes time translation invariance in the transformed space, such that each Wilson line can be considered as a static charge. In addition, the dilatation operator in Minkowski space (whose eigenvalues are dimensions) maps to the operator ∂_τ in radial coordinates, where τ is the time-like coordinate normal to the (space-like) slices of AdS. Eigenvalues of the latter operator are energies (up to a factor of i), so that one may obtain the anomalous dimension associated with a given Wilson line correlator by evaluating the energy of a collection of static charges in Euclidean AdS space. This is worked out in detail in [48] for QED and QCD at one loop, and the conceptual picture thus obtained is used to motivate a family of *conformal gauges* for use in field theory Wilson line calculations. At two loops, as shown explicitly in [48], such gauges eliminate non-Abelian graphs containing a three-gluon vertex. Although such graphs are not completely absent at higher orders, it seems likely that such gauges may provide a natural framework for probing the conjectured all-order structure of IR singularities in QCD fixed-angle scattering amplitudes [30, 32, 33], which involves correlations between only pairs of partons in the case that they are all massless.

The purpose of this paper is to extend the radial picture for Wilson lines to perturbative quantum gravity, in the form of General Relativity (GR) minimally coupled to (scalar) matter. The consideration of infrared singularities in gravity dates back to [49], and there has recently been renewed interest [50–52], based on a number of motivations. Firstly, although GR contains non-renormalisable ultraviolet singularities, alternative field theories of gravity may have the same long distance behaviour [53]. Secondly, there is a growing body of work involving intriguing connections between scattering amplitudes in gauge and gravity theories, at arbitrary loop level (see e.g. [54–56], and [57, 58] for recent applications), which may have a string theoretic origin. There then exists the real possibility that knowledge about quarks and gluons can tell us about gravity, or vice versa. To this end, it is important to develop common conceptual pictures, that allow us interpret similar physics (such as infrared singularity structures) in a natural way, an observation which motivates the present study.

The structure of the paper is as follows. In section 2 we review the radial picture for Wilson lines developed in [48], as well as recent results from perturbative quantum gravity, which will be necessary for the rest of the paper. In section 3, we apply the radial picture to Wilson line operators in quantum gravity, showing explicitly that the form of the one-loop cusp anomalous dimension in gravity can be reproduced from the Newtonian limit of General Relativity in Euclidean AdS space. In section 4, we comment on the lightlike limit of gravitational Wilson line operators, in particular observing the cancellation of collinear singularities in the radial picture. In section 5 we introduce a general formulation of the cusp anomalous dimension calculation, and show that a continuous parameter exists that interpolates smoothly between the QED and gravity cases. In section 6, we discuss our results before concluding. Some technical details are collected in appendices.

2 Review of necessary concepts

2.1 The radial picture for Wilson lines

In this section, we briefly review the approach of [48] for describing Wilson lines as static charges in AdS space. Given that we will be focussing explicitly on gravity in the rest of the paper, we here discuss only QED, thus ignoring additional complications due to non-trivial colour structure.

Let us start with the definition of a Wilson line operator in an Abelian gauge theory:

$$\mathcal{W}(\mathcal{C}) = \exp\left[ie \int_{\mathcal{C}} dx_\mu A^\mu(x)\right], \tag{1}$$

where $A^\mu(x)$ is the gauge field, e the coupling constant, and the line integral is taken over the contour $\mathcal{C}$ in Minkowski space. In applications of Wilson lines to scattering amplitudes, one is interested in Wilson line contours which are fully determined by the momenta of the outgoing particles. Each Wilson line then represents an outgoing particle dressed by an infinite number of soft photon emissions, and the soft part of the scattering amplitude is given by the vacuum expectation value of the product

$$\mathcal{W}(n_1, \ldots, n_L) = \prod_{i=1}^{L} \exp\left[ie \int_0^\infty ds_i p_i \cdot A(sp_i)\right], \tag{2}$$

where L is the number of hard outgoing particles. Here we have parameterised each (straight) Wilson line contour by $x_i = s_i p_i$, where p_i is the 4-momentum of outgoing particle i. Note that s_i then has dimensions of (length)^2. It is more common in the literature to instead write $x_i = t_i n_i$, where n_i is the 4-velocity. The parameter t_i then has dimensions of length, but one is free to make the above choice, given that the Wilson line operator is invariant under rescalings

$$p_i \rightarrow \frac{p_i}{\lambda}, \quad s_i \rightarrow \lambda s_i. \tag{3}$$

The fact that we choose to use the momentum rather than the velocity in eq. (1) is to make explicit the analogy between gauge theory and gravity, where in the latter theory momenta are more important given that they play the role of charges.

Consider now Minkowski space with time coordinate t, and spherical polar coordinates r, θ and ϕ, representing the radial, polar and azimuthal coordinates respectively. Then a given Wilson line direction can be parametrised by [48]

$$x^\mu = e^\tau(\cosh\beta, \sinh\beta\,\hat{\boldsymbol{n}}), \tag{4}$$

where $\hat{n}$ is a unit 3-vector. In these so-called *radial coordinates*, the Minkowski space metric becomes[5]

$$ds^2 = e^{2\tau}\left[d\tau^2 - (d\beta^2 + \sinh^2\beta\, d\Omega_2^2)\right], \tag{5}$$

where $d\Omega_2^2$ is the squared line element on a 2-sphere. Next, one interprets $-\infty \leq \tau \leq \infty$ as a time coordinate, and uses the fact that Abelian gauge theory (in the absence of propagating fermions) is

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[5]Note that in this subsection only we use the metric $(+,-,-,-)$, for ease of comparison with [48]. Throughout the rest of the paper, we will use the alternative choice $(-,+,+,+)$, which is more common in the gravity literature.

classically conformally invariant in four dimensions. That is, one is free to rescale the line element of eq. (5) by an overall factor, and thus to replace eq. (5) by

$$ds^2 = d\tau^2 - (d\beta^2 + \sinh^2 \beta \, d\Omega_2^2). \tag{6}$$

The time coordinate is now explicitly decoupled from the spatial coordinates, and spatial slices (i.e. at fixed τ) constitute Euclidean AdS space in three dimensions. Two important identifications between the original Minkowski space and the transformed space are as follows:

- Reparametrisations of the Wilson line contour in Minkowski space, e.g. the rescalings of eq. (3), map to time translations in the transformed space. Thus, reparametrisation invariance of Wilson lines shows up, in the latter space, as invariance under shifts in τ. This allows one to consider the Wilson lines in the transformed space as static charges.

- The dilatation operator $x^\mu \partial_\mu$ (whose eigenvalues are dimensions) maps to the operator ∂_τ in the transformed space. This is related to the Hamiltonian operator in the transformed space via

$$\partial_\tau = i\mathcal{H}^{\mathbb{R} \times \text{AdS}}, \tag{7}$$

so that the anomalous dimension of a Wilson line correlator maps to the energy of the corresponding collection of static charges in the Euclidean AdS space, with an additional factor of i.

The electrostatic potential ϕ due to the two charges is given by the solution of Laplace's equation in Euclidean AdS space:

$$\nabla^2 \phi = \frac{1}{\sinh^2 \beta} \partial_\beta \left(\sinh^2 \beta (\partial_\beta \phi) \right) = 0, \tag{8}$$

whose solution is

$$\phi(\beta) = C_1 + C_2 \coth \beta, \tag{9}$$

where the $\{C_i\}$ are constants of integration. As explained in detail in [48], this does not have the correct behaviour as $\beta \to \infty$, where the cusp anomalous dimension should diverge linearly in β, due to the appearance of collinear singularities in Minkowski space. This is due to the fact that the solution implicitly includes the effect of a spurious charge corresponding to a phantom initial state particle (i.e. eq. (9) has a pole at $\beta = 0$ and $\beta = i\pi$). One may correct for this by first analytically continuing the spacelike part of the radial coordinate space to a Euclidean 3-sphere, and subsequently adding a constant charge density to the source term of the equation for the electrostatic potential. These constant densities cancel out upon constructing any collection of Wilson line operators, which must satisfy charge conservation. The corrected solution is

$$\phi = \frac{1}{4\pi^2} \left[(\pi + i\beta) \coth \beta + C \right], \tag{10}$$

where the overall normalisation is fixed by the amount of constant charge necessary to cancel the phantom charges.

Having constructed the potential, the total energy of a pair of charges (which in [48] is obtained by first constructing the electric field) is

$$E(\beta_{12}) = \frac{q_1 q_2}{4\pi^2} \left[(\pi + i\beta_{12}) \coth \beta_{12} + C \right], \tag{11}$$

5

where β_{12} is the geodesic distance between the charges, which is identified with the cusp angle in Minkowski space. The constant C can be fixed by analytic continuation to the situation in which one of the Wilson lines is incoming. The vanishing of the cusp anomalous dimension when the two Wilson lines become parallel then implies that $C = -i$ [48]. Finally, one finds that the cusp anomalous dimension should be given by

$$\Gamma = iE(\beta_{12}) = \frac{q_1 q_2}{4\pi^2}\left[(i\pi - \beta_{12})\coth\beta_{12} + 1\right],\tag{12}$$

in exact agreement with an explicit Minkowski space field theory calculation. The above argument is for the case of QED. The generalisation to a non-Abelian context is straightforward, and is also presented in [48].

2.2 Infrared singularities in Quantum GR

In the previous section, we have reviewed the arguments of [48] which relate the cusp anomalous dimension in a gauge theory to the energy of a static charge configuration in Euclidean AdS space. The aim of the present paper is to outline how the same arguments can be applied in quantum gravity. To this end, we must first recap salient features regarding the structure of infrared singularities in GR. For ease of comparison with previous literature on quantum gravity, we will from now on adopt the $(-,+,+,+)$ metric, in contrast with the previous subsection.

Here we will consider general relativity minimally coupled to a scalar field Φ, a theory whose total action is given by

$$S = S_{\text{E.H.}}[g^{\mu\nu}] + S_{\text{mat}}[\Phi^*, \Phi, g^{\mu\nu}],\tag{13}$$

where $S_{\text{E.H.}}$ is the Einstein-Hilbert action, and

$$S_{\text{mat}}[\Phi^*, \Phi, g^{\mu\nu}] = \int d^d x \sqrt{-g}\left[-g^{\mu\nu}\partial_\mu\Phi^*\partial_\nu\Phi - m^2\Phi^*\Phi\right]\tag{14}$$

in d dimensions, where g is the determinant of the metric tensor $g_{\mu\nu}$. Perturbation theory can be defined by expanding the metric tensor according to

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu},\tag{15}$$

where $h_{\mu\nu}$ is then the graviton field, and $\kappa = \sqrt{16\pi G_N}$, with G_N Newton's constant. Note that a different choice is often made in the literature (see e.g. [59]), in which one instead defines the graviton via the expansion of the quantity

$$\tilde{g}^{\mu\nu} = \sqrt{-g}g^{\mu\nu}.\tag{16}$$

This simplifies the Feynman rules for emission of gravitons from scalar particles, and was the choice adopted by [51] which derived Wilson line operators for soft graviton emission. Here, however, we will stick with the choice of eq. (15), for reasons that will become clear. Substituting eq. (15) into eq. (14) and expanding up to first order in κ (including also the factor $\sqrt{-g}$), the matter action

becomes[6]

$$S_{\text{mat}}[\Phi^*, \Phi, h^{\mu\nu}] = \int d^d x \left[-\partial^\mu \Phi^* \partial_\mu \Phi - m^2 \Phi^* \Phi + \kappa \left(-\frac{m^2}{2} \eta^{\mu\nu} \Phi^* \Phi - \frac{1}{2} \eta^{\mu\nu} \partial_\alpha \Phi^* \partial^\alpha \Phi \right. \right.$$
$$\left. \left. - \partial^\mu \Phi^* \partial^\nu \Phi \right) h_{\mu\nu} \right] + \mathcal{O}(\kappa^2). \tag{17}$$

The momentum-space Feynman rule for single graviton emission from a scalar is then

$$\frac{i\kappa}{2} \left[(-m^2 + p_1 \cdot p_2) \eta^{\mu\nu} + p_1^\mu p_2^\nu + p_1^\nu p_2^\mu \right], \tag{18}$$

where p_1 and p_2 are both outgoing. We will also need the graviton propagator, which in d dimensions is given (in the de Donder gauge) by [60]

$$D_{\mu\nu,\alpha\beta}(k) = \frac{-iP_{\mu\nu,\alpha\beta}}{k^2 - i\epsilon}, \quad P_{\mu\nu,\alpha\beta} = \eta_{\mu\alpha} \eta_{\nu\beta} + \eta_{\mu\beta} \eta_{\nu\alpha} - \frac{2}{d-2} \eta_{\mu\nu} \eta_{\alpha\beta}. \tag{19}$$

Given the theory defined by eq. (13), one may consider scattering amplitudes involving a number L of external scalars, which interact by the exchange of gravitons. When emitted gravitons become soft, infrared divergences occur. The study of infrared singularities in quantum general relativity dates back to the classic paper [49], which established that Abelian exponentiation can be generalised to gravity from the QED case. Earlier this year, attention again turned to the IR sector of gravity, with a particular emphasis on describing the singularity structure using methods developed in the context of gauge theory [50–52], leading both to an extension of our knowledge of IR effects in gravity, and also a more unified view of the shared physics underlying the IR sectors of gravity and gauge theories. It has been known for some years that amplitudes in gauge theory with L external lines factorise into the schematic form [61–65]

$$\mathcal{A}_L = \mathcal{S} \cdot \mathcal{H} \cdot \prod_{i=1}^{L} J_i. \tag{20}$$

Here $\mathcal{S}$ is a *soft function*, collecting all infrared singularities due to soft gauge boson emission, $\mathcal{H}$ is a *hard function* which is finite in dimensional regularisation as $d \to 4$, and J_i is a *jet function* collecting hard collinear singularities associated with the ith external line[7]. The soft function is given by a vacuum expectation value of Wilson lines meeting at a point, where there is one Wilson line for each external particle. The authors of [50] suggested that a similar structure should hold for pure quantum gravity, namely that one has

$$\mathcal{A}_L = \mathcal{S} \cdot \mathcal{H} \tag{21}$$

in the gravitational context, where again $\mathcal{S}$ and $\mathcal{H}$ denote soft and hard functions, but in this case there are no jet functions, due to the fact that collinear singularities cancel after summing over diagrams. The latter fact was first established in [49], for soft-collinear emissions. As in the

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[6]Note that after expanding in the weak field approximation, contractions involving upper and lower indices are interpreted as involving the Minkowski metric $\eta_{\mu\nu}$.

[7]Strictly speaking, there is a double counting in eq. (20), given that soft-collinear singularities are present in both the jet functions and the soft function. This can be easily rectified by dividing by *eikonal jet functions* $\mathcal{J}_i$, which we may ignore in the present discussion.
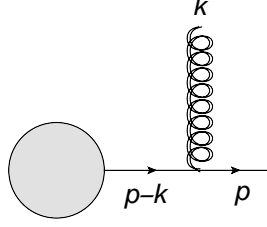
Figure 1: The emission of a graviton from an external scalar line.

gauge theory case, the soft function is given by a Wilson line correlator, where an appropriate form of the Wilson line operator (representing the emission of soft gravitons from an eikonal line) was suggested in [50][8]. Furthermore, the anomalous dimension governing the renormalization of gravitational Wilson line correlators was argued to be one loop exact. Reference [51] investigated some of these issues further in the context of GR minimally coupled to matter (as described by eq. (13)), and derived explicit forms for the Wilson line operators of quantum gravity using path integral techniques borrowed from gauge theory [24]. This also generalised the discussion to include the effects of massive eikonal lines, and corrections beyond the eikonal approximation. Finally, the authors of [52] have undertaken a complete and rigorous analysis of the infrared singularity structure of pure gravity amplitudes, using techniques earlier applied to QCD (see [66] for a pedagogical review). This confirmed the one-loop exactness of the anomalous dimension underlying the IR singularity structure of gravity amplitudes. In addition, the cancellation of collinear singularities was generalised from soft to hard collinear configurations.

For this paper, we will need the form of the gravitational Wilson line operator for the weak field expansion of eq. (15), for a straight line contour along the direction of an outgoing particle. This is given by [9]

$$
\mathcal{W}_g = \exp\left[\frac{i\kappa}{2} \int_0^\infty ds\, p^\mu p^\nu\, h_{\mu\nu}(sp^\mu)\right], \tag{22}
$$

for momentum p^μ. Note that this is not the same as the result presented in [51], owing to the fact that that paper used a weak field expansion based on eq. (16). However, we can justify this result as follows. First, note that the emission of a graviton from an external scalar line is given by figure 1. This introduces a propagator for the intermediate scalar line, and an emission vertex for the graviton. The latter is given by the vertex of eq. (18) with $p_1 = -(p - k)$ and $p_2 = p$ (recalling that the $\{p_i\}$ were defined to be outgoing above), which in the soft limit $k \to 0$ becomes

$$
\frac{i\kappa}{2}\left[-(p^2 + m^2)\eta^{\mu\nu} - 2p^\mu p^\nu\right] = -i\kappa\, p^\mu p^\nu. \tag{23}
$$

Combining this with the propagator for the intermediate scalar line gives

$$
\frac{-i}{(p - k)^2 + m^2 - i\epsilon}(-i\kappa\, p^\mu p^\nu) = \frac{\kappa}{2}\frac{p^\mu p^\nu}{p \cdot k}. \tag{24}
$$

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[8]Note that what we refer to as a Wilson line in this paper is not to be confused with the parallel transport operator of GR, defined in terms of the Christoffel symbol, which is also sometimes referred to in Wilson line terms.

[9]An alternative Wilson line operator was studied in [67]. See also [50] for a discussion of this point.

The right-hand side constitutes an effective Feynman rule for the emission of a soft graviton from an eikonal line. In particular, it is insensitive to the spin of the emitting particle. This same Feynman rule is indeed generated by the Wilson line operator of eq. (22), which shows that the latter is the correct operator.

An interesting consistency check of the above Wilson line operator is that one may derive Newton's law of gravity, by calculating the expectation value of a certain Wilson loop in Minkowski space. This is analogous to the derivation of the Coulomb potential in QED. Although the derivation of Newton's law from perturbative quantum gravity is by no means a new result [60], this calculation is usually performed by taking the non-relativistic limit of the full one-graviton exchange graph. Given that the alternative derivation gives extra insight into the gravitational Wilson line operators, we present this here in appendix A[10].

Armed with the above results, we are now ready to examine Wilson line operators in quantum gravity, using the Euclidean AdS picture of [48]. This is the subject of the following section.

3   Gravitational Wilson lines in the radial picture

In the previous sections, we have reviewed both the radial coordinate picture for calculating cusp anomalous dimensions of Wilson line operators, and also the known structure of infrared divergences in quantum GR. In this section, we combine these concepts, in order to elucidate the gravitational analogue of the analysis of [48] for abelian and non-abelian gauge theories. As stated in the previous section, we use the metric $(-,+,+,+)$, so that the equivalent of the radial coordinate space of eq. (5) is

$$ds^2 = e^{2\tau}[-d\tau^2 + (d\beta^2 + \sinh^2 \beta \, d\Omega_2^2)]. \tag{25}$$

Our starting point is to note that, unlike the case of conventional gauge theories, the Wilson line operator of eq. (22) is not invariant under rescalings of the form of eq. (3). This does not prevent us from examining Wilson lines in radial coordinates. However, it is not then true that the system in the transformed space will be invariant under time translations (in τ). In particular, one must include the prefactor of $e^{2\tau}$ in the metric of eq. (25), which changes the nature of computations involving energies in radial space. In calculating the latter, we will be interested in the component $h_{\tau\tau}$ of the graviton field. The lack of rescaling invariance in Minkowski space implies that this will be τ-dependent. We can surmise this dependence by rewriting the Wilson line operator of eq. (22) as

$$\mathcal{W}_g = \exp\left[\frac{i\kappa}{2}p^\mu \int_0^\infty ds\, p^\nu\, h_{\mu\nu}(sp^\mu)\right], \tag{26}$$

where symmetry of the graviton field ($h_{\mu\nu} = h_{\nu\mu}$) means that we can decide to take either of the momentum factors out of the integral along the Wilson line parameter. The exponent now has the explicit form of a reparametrisation-invariant term

$$\int dx^\nu\, h_{\mu\nu}(x^\mu) \tag{27}$$

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[10]This calculation is closely related to a similar analysis in [68], as we explain in the appendix.

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multiplied by a charge factor p^μ. That this charge is the 4-momentum of the emitting particle is as expected from general relativity. Rewriting the expression (27) as

$$\int_0^\infty d\tau \, \dot{x}^\nu \, h_{\mu\nu}(x^\mu), \qquad (28)$$

where the dot represents differentiation with respect to τ, it follows that

$$\frac{\partial}{\partial \tau} \dot{x}^\nu h_{\mu\nu} = \frac{\partial}{\partial \tau} h_{\mu\tau} = 0, \qquad (29)$$

so that the invariance under τ translations results from the invariance of the expression (28) under reparametrisations. That is, the quantity $h_{\tau\nu}$ is independent of τ in the radial coordinate space [11]. As a consequence, one may write

$$h_{\tau\tau} = \dot{x}^\mu \dot{x}^\nu h_{\mu\nu} = e^\tau \, K(\beta), \qquad (30)$$

where we have used the fact that $\dot{x}^\mu = x^\mu = e^\tau n^\mu$, with n^μ independent of τ. Given that n^μ is in the direction of the hard momentum p^μ of the particle emitting soft gravitons, we may write

$$n^\mu = \frac{p^\mu}{m}, \quad x^\mu = e^\tau \frac{p^\mu}{m}, \qquad (31)$$

where the normalisation ensures $n^2 = -1$.

Analogously to the QED analysis of [48], one may consider calculating the potential energy of a pair of charges in the radial coordinate space. In this case, these will be masses, which we label by m_1 and m_2. The total potential energy can be constructed by first considering the potential energy generated by the mass m_2 located at the origin of the radial coordinate space, and then considering the mass m_1 as a test particle a geodesic distance β away.

To this end, we must construct the equation satisfied by $h_{\tau\tau}$ in radial coordinate space, and the first step is to note that the Wilson line phase of eq. (26) can be written as

$$\frac{i\kappa}{2} p^\nu \int_0^\infty dx^\mu \, h_{\mu\nu}(sp^\mu) = \frac{i\kappa}{2} p^\nu \int_{-\infty}^\infty d\tau \, h_{\tau\nu}. \qquad (32)$$

Inserting a delta-function in the radial coordinate space, this can be further rewritten as

$$\frac{i\kappa}{2} p^\nu \int_{-\infty}^\infty d\tau \int d^3\boldsymbol{x} \, \delta^{(3)}(\boldsymbol{x}) \, h_{\tau\nu}. \qquad (33)$$

Equating this to the general form of a source term in curved space,

$$i \int d^4x \sqrt{-g} j^{\mu\nu} h_{\mu\nu},$$

one finds that the current which sources the conformally invariant quantity $h_{\tau\nu}$ is given by

$$\sqrt{-g} j^{\tau\nu} = \frac{\kappa}{2} p^\nu \, \delta^{(3)}(\boldsymbol{x}). \qquad (34)$$

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[11] Care is needed here in interpreting the notation. The index ν in $h_{\tau\nu}$ is a Minkowski space index, which has yet to be transformed to the radial coordinate space.

One then finds

$$j_{\mu\tau} = g_{\tau\tau} j_\mu^\tau = -\frac{\kappa}{2} e^{-2\tau} p_\mu \delta^{(3)}(\boldsymbol{x}), \tag{35}$$

where we have absorbed geometric factors into the delta function so that this is now normalised according to

$$\int d^3\boldsymbol{x} \sqrt{g^{(3)}} \delta^{(3)}(\boldsymbol{x}) = 1, \tag{36}$$

where $g^{(3)}$ is the determinant of the spatial part of the metric $g_{\mu\nu}$, disregarding the overall factor of $e^{2\tau}$. The component $h_{\tau\tau}$ of the graviton field satisfies the wave equation (see e.g. [69])

$$\Box h_{\tau\tau} = j_{\tau\tau}, \tag{37}$$

where from eqs. (31, 35) one finds

$$j_{\tau\tau} = \dot{x}^\mu j_{\mu\tau} = \frac{\kappa}{2} m_2 e^{-\tau} \delta^{(3)}(\boldsymbol{x}). \tag{38}$$

The left-hand side of eq. (37) is the covariant D'Alambertian operator

$$\Box = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} g^{\mu\nu} \partial_\nu \right). \tag{39}$$

We see from eq. (37) that gravitational Wilson lines in the radial space correspond to time-dependent sources. This is a manifestation of the overall $e^{2\tau}$ factor in the metric of eq. (25): the radial size of the spacelike part of the space changes with τ. If we consider a mass fixed at the origin, the flux of the gravitational field through an expanding surface of radius $\sim e^\tau \beta$ will be constant. If we instead consider a surface at fixed β around the origin, the flux through such a surface grows as a function of time, such that it looks as if a mass $m e^{-\tau}$ is enclosed[12], which is precisely the content of eq. (37).

Using the metric of eq. (25), eq. (37) becomes

$$e^{-2\tau} \left(\nabla^2 - 3 \right) h_{\tau\tau} = \frac{\kappa}{2} m_2 e^{-\tau} \delta^{(3)}(\boldsymbol{x}), \tag{40}$$

where ∇^2 is the three-dimensional Laplacian[13]. The homogeneous equation is not simply Laplace's equation as in the case of QED, but rather the Helmholtz equation. The additional term linear in $h_{\tau\tau}$ is a direct consequence of the fact that the flux through a surface of fixed β is changing with time due to the overall time dependence of the metric, as discussed above. The relation of the graviton field to the Newtonian potential is (again see e.g. [69])

$$\Phi = \frac{\kappa}{2} h_{\tau\tau}, \tag{41}$$

so that eq. (40) may be rewritten as

$$\nabla^2 \Phi - 3\Phi = \left(\frac{\kappa}{2} \right)^2 m_2 e^\tau \delta^{(3)}(\boldsymbol{x}). \tag{42}$$

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[12]One must also remember that there is a factor $\sqrt{-g}$ involved in calculating the flux, so that a mass which decreases with time is consistent with a flux which grows with time.

[13]Note that, for ease of comparison with the QED case, we denote by ∇^2 the Laplacian without the τ dependence implied by the metric of eq. (25).

This is the gravitational analogue of Laplace's equation for the QED case of [48]. It is essentially Newton's law of gravity, but with an extra term proportional to Φ resulting from the use of a time-dependent curvilinear coordinate system. From eqs. (30, 41) we may write

$$\Phi(\beta, t) = \tilde{K}(\beta)e^{\tau}, \quad \tilde{K}(\beta) = \frac{\kappa}{2}K(\beta). \tag{43}$$

Then using the Laplacian of eq. (8), we must find the general solution of the homogeneous equation

$$\frac{1}{\sinh^2 \beta}\partial_\beta \left(\sinh^2 \beta \partial_\beta \tilde{K}(\beta)\right) - 3\tilde{K}(\beta) = 0, \tag{44}$$

which is

$$\tilde{K}(\beta) = A_1 \left[\frac{1}{\sinh \beta} + 2\sinh \beta\right] + A_2 \cosh \beta. \tag{45}$$

Note that, as in the QED case, the general solution of the homogeneous equation is a superposition of a function which is even under the transformation of eq. (50), and a function which is odd. These are the first and second terms in eq. (45) respectively. As shown in appendix B, one can fix A_1 by considering a 3-surface consisting of a cylinder of radius β_0 and height (in time) τ_0. The result is

$$\tilde{K}(\beta) = m_2 \frac{\kappa^2}{16\pi} \left[\frac{1}{\sinh \beta} + 2\sinh \beta\right] + A_2 \cosh \beta. \tag{46}$$

Another feature shared with the QED case is that this solution has the wrong behaviour as $\beta \to \infty$. As shown in appendix C, the actual result $\sim \beta e^{\beta}$ at large β. This is a factor e^{β} up on the QED case (where the divergence is linear), due to the fact that there are extra momentum factors in gravity. The failure of the solution (46) to reproduce this behaviour is, as in the QED case, due to a spurious charge. Noting that eq. (46) has a pole at both $\beta = 0$ and $\beta = i\pi$, this solution represents the combined effect of the physical mass at $\beta = 0$, and a spurious mass at $\beta = i\pi$.

In [48], and as reviewed in section 2, this problem is solved in the QED case by modifying the current j_τ by a constant charge density. A similar procedure can be used in the gravity case, as we now discuss. First, recall that the current $j_{\mu\tau}$ is given by eq. (35), and that this sources the *conformally invariant* quantity $h_{\mu\tau}$ involving one Minkowski-space and one radial-space index. If we were to solve for $h_{\mu\tau}$, we would find a charge p^μ at the origin $\beta = 0$ of the radial space, and a spurious charge $-p^\mu$ (corresponding to an incoming momentum in Minkowski space) at $\beta = i\pi$. The analogue of adding a constant charge density in the present case is to modify the current of eq. (35) so as to give

$$j_{\mu\tau} = -\frac{\kappa}{2}e^{-2\tau} p_\mu \left[\delta^{(3)}(\boldsymbol{x}) + K\right], \tag{47}$$

where K is such as to remove the spurious incoming momentum. Any given collection of gravitational Wilson lines will obey momentum conservation, such that the constant charge densities thus added will cancel out. This is the analogue of the cancellation of the constant terms due to electric charge conservation in the QED case.

Above, we solved directly for $h_{\tau\tau}$, which is related to the Newtonian potential in the radial coordinate space. To this end, one must consider the modified current

$$j_{\tau\tau} = \dot{x}^\mu j_{\mu\tau} = \frac{\kappa}{2}e^{-2\tau}\dot{x}^\mu p_\mu \left[\delta^{(3)}(\boldsymbol{x}) + K\right]. \tag{48}$$

Here $\dot{x}^\mu = x^\mu = e^\tau (\cosh\beta, \sinh\beta \boldsymbol{n})$ is a general point in the radial coordinate space. In the first term, this is constrained to be related to the momentum p^μ (corresponding the test charge at the origin) by eq. (31), due to the delta function. In the second term this is not the case, and using $p^\mu = m_2(1, \boldsymbol{0})$ (in Minkowski coordinates) one finds

$$j_{\tau\tau} = \frac{\kappa}{2} e^{-\tau} \left[m_2 \delta^{(3)}(\boldsymbol{x}) + K \cosh\beta \right]. \tag{49}$$

That is, the effect of a constant current density for the conformally invariant quantity $h_{\mu\tau}$ becomes a non-trivial charge density $\sim \cosh\beta$ in the current for $h_{\tau\tau}$. That this makes physical sense can be seen as follows. We have already noted that the spurious pole of eq. (46) corresponds to a mass at $\beta = i\pi$, in addition to the mass at $\beta = 0$. Both of these masses are necessarily positive. However, the charge density $\cosh\beta$ has the property of being *odd* under the transformation

$$\beta \to i\pi - \beta. \tag{50}$$

Thus, the added charge density reinforces the physical charge in the upper branch of the AdS space, but acts to cancel out the fake charge in the lower branch, as required. Absorbing various factors in the constant charge density K, eq. (44) is modified to

$$\frac{1}{\sinh^2\beta} \partial_\beta \left(\sinh^2\beta \partial_\beta \tilde{K}(\beta) \right) - 3\tilde{K}(\beta) = B\cosh\beta, \tag{51}$$

whose general solution is

$$\tilde{K}(\beta) = (A_1 + A_3\beta) \left[\frac{1}{\sinh\beta} + 2\sinh\beta \right] + A_2 \cosh\beta, \tag{52}$$

where $A_3 = B/8$ is to be determined, and A_1 is given by eq. (107). One can fix A_3 using the requirement that the potential must not diverge at $\beta = i\pi$, if the spurious charge has been consistently removed:

$$A_3 = -\frac{A_1}{i\pi}. \tag{53}$$

The solution of the Newtonian potential which respects all boundary conditions is therefore

$$\Phi(\beta, \tau) = m_2 e^\tau \left[\frac{\kappa^2}{16\pi^2} (i\beta + \pi) \left[\frac{1}{\sinh\beta} + 2\sinh\beta \right] + C\cosh\beta \right], \tag{54}$$

where $C = A_2/m_2$ plays the same role as the constant in the QED case. We see that the time-dependent potential contains the combination $m_2 e^\tau$, as a consequence of the same combination occuring on the right-hand side of eq. (42). We can thus identify a static potential

$$\tilde{\Phi}(\beta) = m_2 \left[\frac{\kappa^2}{16\pi^2} (i\beta + \pi) \left[\frac{1}{\sinh\beta} + 2\sinh\beta \right] + C\cosh\beta \right]. \tag{55}$$

This is correct at all times if we consider m_2 to be time-dependent ($m_2(\tau) \equiv m_2 e^\tau$, where m_2 is the static mass), which takes into account the fact that gravitational Wilson lines are not invariant under reparametrisations of p^μ.

Now consider adding a test particle of mass m_1 at location β. If m_2 is at $\beta = 0$, then $\beta \equiv \beta_{12}$ is the geodesic separation between the masses in the AdS space. The total time-independent potential is then given by

$$E(\beta_{12}) = m_1 \tilde{\Phi} = m_1 m_2 \left[\frac{\kappa^2}{16\pi^2}(i\beta_{12} + \pi) \left[\frac{1}{\sinh \beta_{12}} + 2\sinh \beta_{12} \right] + C \cosh \beta_{12} \right]. \tag{56}$$

Note that the potential energy is defined only up to an arbitrary amount of the solution to the homogeneous equation[14]. In the present case, this includes a contribution proportional to $\cosh \beta$, as seen explicitly in eq. (46). The constant C can be fixed, as in the QED case, by analytically continuing eq. (56) to the case of one incoming and one outgoing particle, corresponding to the transformation of eq. (50) for the test particle of mass m_1. Then

$$E(\beta_{12}) \rightarrow -im_1 m_2 \left[\frac{\kappa^2}{16\pi^2}\beta_{12} \left[\frac{1}{\sinh \beta_{12}} + 2\sinh \beta_{12} \right] - iC \cosh \beta_{12} \right]. \tag{57}$$

This must vanish as $\beta \rightarrow 0$, corresponding to the fact (in the original Minkowski space) that the anomalous dimension vanishes for a straight Wilson line contour with no cusp. This fixes

$$C = -\frac{i\kappa^2}{16\pi^2}, \tag{58}$$

so that the potential energy in the original setup of two final state Wilson lines is

$$E(\beta_{12}) = m_1 m_2 \frac{\kappa^2}{16\pi^2} \left[(i\beta_{12} + \pi) \left(\frac{1}{\sinh \beta_{12}} + 2\sinh \beta_{12} \right) - i \cosh \beta_{12} \right]. \tag{59}$$

Finally, the cusp anomalous dimension in Minkowski space must be given by

$$\Gamma = iE = m_1 m_2 \frac{\kappa^2}{16\pi^2} \left[(i\pi - \beta_{12}) \left(\frac{1}{\sinh \beta_{12}} + 2\sinh \beta_{12} \right) + \cosh \beta_{12} \right]. \tag{60}$$

This agrees exactly with the field theory calculation of this quantity in Minkowski space, which we present here in appendix C.

If many different masses are present, one finds a total anomalous dimension

$$\Gamma_{tot} = \sum_{i<j} \frac{\kappa^2}{16\pi^2} m_i m_j \left[(i\pi - \beta_{ij}) \left(\frac{1}{\sinh \beta_{ij}} + 2\sinh \beta_{ij} \right) + \cosh \beta_{ij} \right], \tag{61}$$

corresponding to the total energy in the radial coordinate space. The sum is over all distinct pairs of outgoing particles, such that the cusp angle in each case is given by

$$\cosh \beta_{ij} = -\frac{p_i \cdot p_j}{m_i m_j}. \tag{62}$$

Some further comments are in order regarding the anomalous dimension that we have derived. Firstly, note that eq. (61) is not the most convenient way to express this result, with a view to

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[14]In the QED case, for which Laplace's equation holds, this amounts to the presence of an arbitrary constant.

examining the light-like limit of $\beta_{ij} \to \infty$. Instead, we may pull out an overall factor of $\cosh \beta_{ij}$ in each term to obtain the alternative form

$$\Gamma = -\sum_{i<j} p_i \cdot p_j \frac{\kappa^2}{16\pi^2} \left[(i\pi - \beta_{ij}) \left(2\coth \beta_{ij} - \frac{1}{\sinh \beta_{ij} \cosh \beta_{ij}} \right) + 1 \right], \qquad (63)$$

where we have used the cusp angle definition of eq. (109). This will be useful in the following section.

Secondly, there are potential conceptual issues regarding the gravitational cusp anomalous dimension. This is defined in terms of the ultraviolet renormalisation properties of the vertex at which multiple Wilson lines meet. In calculating any particular diagram contributing to a Wilson line expectation value (and thus contributing to the gravitational soft function), one encounters additional ultraviolet singularities relating in principle to the renormalisation of the masses m_i, and the gravitational coupling constant κ. However, gravity is non-renormalisable, such that the latter contributions are ill-defined, leading to additional operators at each order in perturbation theory that need to be included in Wilson line diagrams[15]. One may then worry that it is not possible to define a gravitational cusp anomalous dimension at all. However, this worry can be seen to be misguided for the following reasons: UV singularities of Wilson lines associated with renormalisation of the cusp correspond to IR singularities in scattering amplitudes, thus must be ultimately separable in a meaningful way from UV singularities associated with Lagrangian parameters. Also, as can be inferred from [50–52], the gravitational cusp anomalous dimension is *one-loop exact*. Thus, the situation in which one might have to worry about separating overlapping UV divergent contributions to the gravitional couplings and to the multieikonal vertex (which potentially occurs only beyond one loop order) never arises. It seems, then, that one-loop exactness of the gravitational anomalous dimension is crucially related to the fact that the IR limit of perturbative general relativity must be UV finite.

4 Gravitational Wilson lines in the lightlike limit

In the previous section, we have obtained the cusp anomalous dimension for perturbative GR via an energy calculation in radial coordinate space, similar to the QED / QCD analysis of [48]. To complete this analysis, it is instructive to examine the lightlike limit $p_i^2 \to 0$. As is well known, additional collinear singularities appear in this limit. In gravity, these singularities are present on a diagram by diagram basis, but cancel after summing over all diagrams [49, 52]. The appearance of collinear singularities in the gauge theory case is associated in the radial coordinate space with a linear divergence of the (imaginary) energy of two static charges as $\beta \to \infty$. A nice comparison was outlined in [48] between this phenomenon and linear confinement - the confinement in this case being that of e.g. quarks within jets.

The aim of this section is to briefly revisit this analysis in the context of the gravity calculation carried out in the previous section, in order to complete the conceptual mapping between the gauge theory and gravity cases. We begin with the form of the anomalous dimension given in eq. (63), in which an explicit factor of $p_i \cdot p_j$ has been pulled out in each term. This expression remains valid as $m_i \to 0$, in which case $\beta_{ij} \to \infty$ for all j. Let us now consider the case that particle i indeed

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[15]For a recent discussion of the difficulty of considering running parameters in quantum gravity, see [70].

becomes massless. From eq. (63), we may isolate all contributions involving particle i as

$$\Gamma_i = -\sum_{j\neq i} p_i \cdot p_j \frac{\kappa^2}{16\pi^2}\left[(i\pi - \beta_{ij})\left(2\coth\beta_{ij} - \frac{1}{\sinh\beta_{ij}\cosh\beta_{ij}}\right) + 1\right], \tag{64}$$

which as $m_i \to 0$ $(\beta_{ij} \to \infty)$ becomes

$$\Gamma_i \to \sum_{j\neq i} p_i \cdot p_j \frac{\kappa^2}{8\pi^2}\beta_{ij}. \tag{65}$$

Examining β_{ij} itself, this is

$$\beta_{ij} = \cosh^{-1}\left(-\frac{p_i \cdot p_j}{m_i m_j}\right) = \log\left(-\frac{p_i \cdot p_j}{m_i m_j}\right) + \mathcal{O}(1). \tag{66}$$

We may rewrite the logarithm on the right-hand side as

$$\log\left(-\frac{p_i \cdot p_j}{m_i m_j}\right) = -\log\left(\frac{m_i}{Q}\right) + \log\left(\frac{-p_i \cdot p_j}{Q m_j}\right) = -\log\left(\frac{m_i}{Q}\right) + \mathcal{O}(1), \tag{67}$$

where Q is an arbitrary momentum scale to keep the arguments dimensionless. Substituting this into eq. (65) gives

$$\Gamma_i \to -\frac{\kappa^2}{8\pi^2}\log\left(\frac{m_i}{Q}\right) p_i \cdot \sum_{j\neq i} p_j. \tag{68}$$

Using momentum conservation

$$\sum_i p_i = 0, \tag{69}$$

the total contribution to the cusp anomalous dimension from particle i is

$$\Gamma_i \to \frac{\kappa^2}{16\pi^2}\log\left(\frac{m_i}{Q}\right) p_i^2 = 0. \tag{70}$$

Thus, collinear singularities do not appear in the gravitational soft function. This is not, of course, a new result. The cancellation of soft collinear singularities has been known since [49], and has been generalised to hard collinear singularities in [52]. The essential physical reason for this cancellation can also be obtained by direct analogy with abelian and non-abelian gauge theory: collinear singularities associated with a given particle depend on its squared charge. In QED this is q^2, where q is the electromagnetic charge. In QCD, this is the quadratic Casimir invariant associated with the representation appropriate to the parton of interest. In gravity, the squared charge is the 4-momentum squared, which is zero if collinear singularities are to be present - which ends up removing them.

Although the above argument is formulated for only one particle becoming massless, it generalises straightforwardly to cases involving more than one massless particle. Note that our reasoning does not tell us that massless particles do not contribute at all to the energy in radial coordinate space. Rather, the diverging term involving the geodesic separation of a given particle from one of its partners (as the former becomes massless) cancels after summing all contributions to the potential

energy, from all the other particles. There are still terms which are $\mathcal{O}(\beta^0)$, which we neglected in the above analysis. It is in principle possible to calculate the total potential energy from interactions with the massless particle by giving the massless particle a small mass m_i, as above, and setting $m_i \to 0$ at the end of the calculation after summing all contributions.

How are we to square this with the fact that, for massive Wilson lines, we derived the energy in the radial coordinate space after taking the Newtonian limit? In this limit, it must be true that a massless particle contributes nothing to the energy of a collection of masses, as it does not gravitate. The resolution of this puzzle is that as $m_i \to 0$, the Newtonian limit is no longer valid, as this relies on being able to define velocities which are much less than the speed of light, which is only possible for massive particles. Instead, as $m_i \to 0$, one must take into account special relativistic corrections.

Having now completed our analysis of gravitational Wilson lines in AdS space, and their analogies with the QED case, it is amusing to note that both theories are in fact special cases of a general formulation, with a continuous relation between them. This is the subject of the following section.

5 General formulation

In the previous sections, we have reviewed the properties of Wilson lines in AdS space, and used an analogous analysis to [48] to examine the properties of the cusp anomalous dimension in perturbative GR. In this section, we point out that one can formulate a general calculation for the cusp anomalous dimension, two special cases of which are QED and gravity. Furthermore, we will show that these cases are continuously related to each other. This perhaps adds an interesting additional way of thinking about the results of [48] and the present paper, which may be of further use.

Consider the operator

$$\mathcal{W}_n = \exp\left[i\lambda\, p^{\mu_1}\, p^{\mu_2} \ldots p^{\mu_n} \int_0^\infty ds\, H_{\mu_1\mu_2\ldots\mu_n}(sp) \right], \tag{71}$$

which is clearly defined for integer n. We may recognise this as a generalisation of the Wilson line operators of eqs. (1, 22), where p^μ is the hard momentum of a particle which emits soft quanta of a spin-n gauge field $H_{\mu_1\ldots\mu_n}$ with coupling constant λ [16]. We have again parametrised the straight-line contour of the Wilson line according to eq. (31), which allows to rewrite eq. (71) as

$$\mathcal{W}_n = \exp\left[i\lambda\, p^{\mu_1}\, p^{\mu_2} \ldots p^{\mu_{n-1}} \int dx^{\mu_n}\, H_{\mu_1\ldots\mu_n}(x) \right]$$
$$= \exp\left[i\lambda\, p^{\mu_1}\, p^{\mu_2} \ldots p^{\mu_{n-1}} \int_{-\infty}^\infty d\tau\, H_{\mu_1\ldots\mu_{n-1}\tau}(x) \right]. \tag{72}$$

We see that the Wilson line phase is the product of a reparametrisation invariant quantity $H_{\mu_1\ldots\mu_{n-1}\tau}$ (which has one radial-space index and $n-1$ Minkowski-space indices) and a charge given by $\lambda p^{\mu_1} \ldots p^{\mu_{n-1}}$. The fact that the former is conformally invariant implies that

$$H_{\tau\tau\ldots\tau} = \dot{x}^{\mu_1}\, \dot{x}^{\mu_2} \ldots \dot{x}^{\mu_{n-1}}\, H_{\mu_1\mu_2\ldots\mu_{n-1}\tau} = e^{(n-1)\tau}\, \tilde{H}(\beta), \tag{73}$$

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[16]The reader may object, regarding how to interpret this operator for $n \geq 3$. We return to this point in what follows.

which is a generalisation of eq. (30). Let us now assume that the homogeneous equation of motion for $H_{\tau\ldots\tau}$ is

$$\Box H_{\tau\ldots\tau} = 0 \tag{74}$$

(as is true for the QED and gravity cases). Implementing the behaviour of eq. (73) and using the covariant D'Alambertian operator of eq. (39), one finds that the spatial part of $H_{\tau\ldots\tau}$ satisfies

$$\left[\nabla^2 - (n^2 - 1)\right]\tilde{H}(\beta) = 0. \tag{75}$$

It is straightforward to verify that this equation reduces to eq. (8) and eq. (44) for the QED ($n = 1$) and gravity ($n = 2$) cases respectively. The general solution of the general equation is given by

$$\tilde{H}(\beta) = A_1\left(\frac{\sinh(n\beta)}{\sinh\beta}\right) + A_2\left(\frac{\cosh(n\beta)}{\sinh\beta}\right), \tag{76}$$

where we have chosen to write this explicitly as the sum of two parts which have a definite parity under the transformation of eq. (50). One may check (using hyperbolic function identities) that this result indeed reproduces the QED and gravity results of eq. (9, 45). However, the solution of eq. (76) (as a function at least) is well-defined for any n. In particular, we may consider n, divorced from the original context of eq. (71), to be a continuous parameter that smoothly interpolates between the two solutions we obtained previously.

In both of the cases considered so far, the solution of eq. (76) did not have the right behaviour as $\beta \to \infty$ to correspond to the cusp anomalous dimension in the relevant field theory. This was rectified by modifying the current density in the inhomogeneous equation for the conformally invariant field component $H_{\mu_1\ldots\mu_{n-1}\tau}$ by a constant density. Introducing a delta function in the radial coordinate space, one may write the Wilson line phase from eq. (72) as

$$i\lambda\, p^{\mu_1}\, p^{\mu_2}\ldots p^{\mu_{n-1}}\int_{-\infty}^{\infty}d\tau\int d^3\boldsymbol{x}\,\delta^{(3)}(\boldsymbol{x})H_{\mu_1\ldots\mu_{n-1}\tau}(x), \tag{77}$$

such that the current density that sources the conformally invariant quantity $H_{\mu_1\ldots\mu_{n-1}\tau}$ is given by

$$\sqrt{-g}\,j^{\mu_1\ldots\mu_{n-1}\tau} = \lambda\, p^{\mu_1}\ldots p^{\mu_{n-1}}\delta^{(3)}(\boldsymbol{x}), \tag{78}$$

from which one finds[17]

$$j_{\mu_1\ldots\mu_{n-1}\tau} = \lambda e^{-2\tau}\, p_{\mu_1}\ldots p_{\mu_{n-1}}\delta^{(3)}(\boldsymbol{x}). \tag{79}$$

The appropriate generalisation of the constant charge density procedure is to modify this current to

$$j_{\mu_1\ldots\mu_{n-1}\tau} = \lambda e^{-2\tau}\, p_{\mu_1}\ldots p_{\mu_{n-1}}\left[\delta^{(3)}(\boldsymbol{x}) + K\right], \tag{80}$$

such that one finds

$$\begin{aligned}
j_{\tau\tau\ldots\tau} &= \dot{x}^{\mu_1}\ldots\dot{x}^{\mu_{n-1}}H_{\mu_1\ldots\mu_{n-1}\tau}\\
&= \lambda e^{(n-3)\tau}m^{n-1}\left[\delta^{(3)}(\boldsymbol{x}) + K\cosh^{(n-1)}\beta\right].
\end{aligned} \tag{81}$$

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[17]Again we have absorbed geometric factors into the delta function, so that this is normalised according to eq. (36).

Thus, the constant charge density becomes a smooth distribution $\sim \cosh^{(n-1)} \beta$ distributed throughout space. As for eq. (76), one may continue n away from integer values.

In the QED and gravity cases, the constant charge density procedure ensured that the energy associated with a pair of charges in AdS space diverged with an overall power of β at large separations (corresponding to collinear singularites in Minkowski space). This became somewhat non-trivial in the gravity example, in which it was crucial that the modification to $j_{\tau\tau}$ went like $\cosh \beta$. For integer values $n \geq 3$, we may note that this property does not generalise. One may verify that

$$\left[\nabla^2 - (n^2 - 1)\right] \left(\frac{(C_1 \beta + C_2) \cosh(n\beta)}{\sinh \beta}\right) = \frac{2 C_1 n \sinh(n\beta)}{\sinh \beta}; \tag{82}$$

$$\left[\nabla^2 - (n^2 - 1)\right] \left(\frac{(C_1 \beta + C_2) \sinh(n\beta)}{\sinh \beta}\right) = \frac{2 C_1 n \cosh(n\beta)}{\sinh \beta}. \tag{83}$$

Thus, a function constructed by modifying the solution of the homogeneous equation (eq. (76)) to include an overall power of β, is not consistent with a modified charge density $\sim \cosh^{(n-1)} \beta$ in general (one may also show that the converse is true). Nevertheless, these results are correct in the QED and gravity cases. For QED, eq. (82) applies, and the resulting (constant) charge density picks out the solution of the homogeneous equation which has odd parity under the transformation of eq. (50). For gravity, eq. (83) produces the required $\cosh \beta$ charge density, which then picks out the even solution for $\tilde{H}(\beta)$.

For $n \geq 3$, it is no longer true that the charge density in $j_{\tau \dots \tau}$ arising from the constant charge prescription is such as to modify the energy by a linear term in β. This presumably means that one cannot interpret the resulting energy as a cusp anomalous dimension of a Wilson line operator. However, this is not at all surprising, as the operator of eq. (71) ceases to be meaningful for $n \geq 3$. If eq. (71) is to be interpreted as the operator describing the emission of soft higher spin gauge bosons, then Lorentz invariance demands that the quantity

$$\sum_n g_n \, p^{\mu_1} \dots p^{\mu_{n-1}}$$

be conserved, where the sum is over all external particles n, and g_n is a constant which may depend on a given hard particle in general (see e.g. chapter 13 of [71]). The $n = 1$ and $n = 2$ cases correspond to electromagnetic charge and 4-momentum conservation respectively. However, for $n \geq 3$ no conserved quantity is possible if non-trivial scattering is to occur, which makes the above argument meaningless if $n \geq 3$ (for integer n). That this is seen directly in the general analysis, in terms of the charge densities not matching up, is interesting.

In this section, we have considered a general formulation of the cusp anomalous dimension calculation in Minkowski space, in which the QED and gravity theories emerge as special cases. Indeed, as discussed above, the general Wilson line operator of eq. (71) is only meaningful (for integer n) for $n = 1, 2$. These are precisely the cases of QED and gravity respectively. Nevertheless, it is amusing to note that one may regard n also as a continuous parameter which smoothly interpolates between the one loop cusp anomalous dimensions of QED and GR. Our main motivation in pointing this out is in case this has any further interpretation or utility, in addition to novelty.

6 Conclusion

In this paper, we have examined gravitational Wilson lines, representing the emission of soft gravitons from an eikonal emitter. We have examined in detail the anomalous dimension which controls the renormalisation of the vertex in a general correlator of such Wilson lines, with a view to generalising the analysis of [48] which relates the calculation of such anomalous dimensions to static energies in Euclidean AdS space.

There are a number of motivations for doing this. To start with, gravitational Wilson line operators and their renormalisation properties have only recently been studied. Given that the cusp anomalous dimension forms such a crucial object in gauge theories, any investigation of its properties in a gravitational context is interesting enough by itself. Our analysis also sheds further light on the radial coordinate picture for gauge theory Wilson lines developed in [48], due to a number of subtleties in the gravitational case. For example, the procedure of adding a constant charge density to obtain the correct boundary conditions in the QED analysis implies a $\cosh\beta$ charge density in the gravity case, which is precisely such as to lead to the correct solution for the anomalous dimension, and also to be consistent with the fact that masses, unlike electric charges, can only be positive.

The similarities and differences between the gravity and QED / QCD radial coordinate pictures may be useful in providing further insight and intuition in current research at the boundary between gauge and gravity theories. One example of this is the property of jet confinement in QCD, which has a particularly intuitive description in the radial coordinate picture, and which is absent in gravity. Another example where the radial coordinate picture might be useful is in thinking about the all-order structure of IR singularities. Radial coordinates were used in [48] to motivate a family of *conformal gauges*, in which the contributions from graphs involving multiple gluon vertices was much reduced (e.g. absent at two loop level). It seems likely that this is related to the conjectured dipole structure of soft singularities in QCD [30, 32, 33, 35]. In gravity there is less immediate motivation for considering such conformal gauges, due to the fact that the Wilson line operator is not conformally invariant, and also that the cusp anomalous dimension is known to be one loop exact. However, it may be that the simple structure of IR divergences in gravity (which is manifestly of dipole form), and the relationship between the radial coordinate pictures in both the gravity and gauge theory cases, can be used to gain further insight into the dipole conjecture.

We also considered a general formulation of the cusp anomalous dimension calculation, in which a smooth parameter arises which smoothly interpolates between the QED and gravity cases. This analysis breaks down for $n \geq 3$ (where n is the number of momentum factors in the Wilson line operator), but this is entirely consistent with the fact that such operators are not physically meaningful due the fact that they lead to conserved higher-rank tensorial charges and thus trivial scattering. It is not immediately obvious what the applications of n considered as a continuous parameter might be, but one might hope that it may lead to further insight on the relationship between QED (or its non-Abelian brother, QCD) and gravity.

In summary, the radial coordinate picture potentially offers novel new insights into both gauge and gravity theories, and the relationships between them. The results of this paper provide a useful addition in this regard.
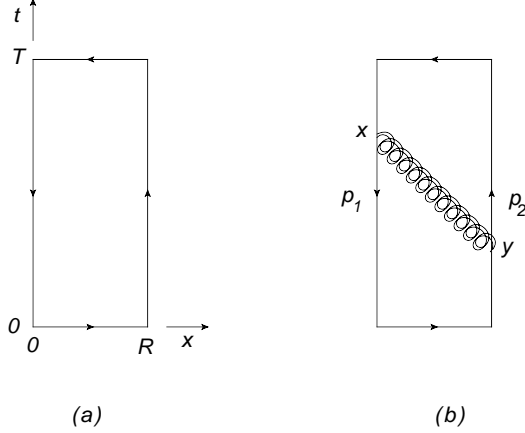
Figure 2: (a) Wilson loop contour in Minkowski space used for the calculation of the Newtonian potential; (b) the relevant one loop diagram.

Acknowlegments

A   Newton's law from the gravitational Wilson line

In this appendix, we show how Newton's law of gravity can be calculated using the Wilson line operator of eq. (22). First, we consider the Minkowski space contour $\mathcal{C}$ of figure 2(a). It is a textbook result in field theory (see e.g. [72]) that the static potential $V(R)$ between two charges of separation R is given by

$$V(R) = \lim_{T \to \infty} \frac{i}{T} \log \langle \mathcal{W}(\mathcal{C}) \rangle, \tag{84}$$

where $\langle \mathcal{W}(\mathcal{C}) \rangle$ is the vacuum expectation value of the Wilson loop along the contour $\mathcal{C}$, and the latter is given in the present case by

$$\mathcal{W}(\mathcal{C}) = \exp\left[i\frac{\kappa}{2} \oint_{\mathcal{C}} ds\, p^\mu\, p^\nu\, h_{\mu\nu}(sp^\mu)\right]. \tag{85}$$

As the time $T \to \infty$, the only contributing diagram at one loop order is that shown in figure 2(b), where the gravitons are emitted at positions

$$x = sp_1, \quad y = tp_2, \tag{86}$$

and the 4-momenta of the static masses are given by

$$p_1 = (m_1, \mathbf{0}), \quad p_2 = (m_2, \mathbf{0}). \tag{87}$$

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One may evaluate this using the position space graviton propagator (in four dimensions)

$$D_{\mu\nu,\alpha\beta}(x-y) = \frac{1}{4\pi^2} \frac{P_{\mu\nu,\alpha\beta}}{(x-y)^2 - i\epsilon}, \tag{88}$$

as can be obtained by Fourier transforming the momentum space propagator of eq. (19). The diagram of figure 2(b) then gives a contribution

$$\log \mathcal{W}_g = \left(\frac{\kappa}{2}\right)^2 \frac{1}{4\pi^2} \int_{T/m_1}^0 ds \int_0^{T/m_2} dt \frac{P_{\mu\nu,\alpha\beta} \, p_1^\mu \, p_1^\nu \, p_2^\alpha \, p_2^\beta}{(x-y)^2 - i\epsilon} + \mathcal{O}(\kappa^4), \tag{89}$$

where we have used the fact that the one-loop contribution to the Wilson loop expectation value is the same as the contribution to the exponent at this order[18]. Using the definition of $P_{\mu\nu,\alpha\beta}$ from eq. (19), one finds

$$P_{\mu\nu,\alpha\beta} \, p_1^\mu \, p_1^\nu \, p_2^\alpha \, p_2^\beta = 2(p_1 \cdot p_2)^2 - p_1^2 p_2^2 = m_1^2 m_2^2. \tag{90}$$

Also transforming the integrals in eq (89) to $x^0 = sm_1$ and $y^0 = sm_2$, one finds

$$\log \mathcal{W}_g = \frac{\kappa^2 m_1 m_2}{16\pi^2} \int_T^0 dx^0 \int_0^T dy_0 \frac{1}{[-(x^0-y^0)^2 + R^2 - i\epsilon]}$$
$$\simeq \frac{\kappa^2 m_1 m_2 T}{16\pi^2} \int_{-T}^T dy_0 \frac{1}{(y^0)^2 - R^2 + i\epsilon}, \tag{91}$$

where in the second line we have used the fact that we are taking $T \to \infty$. The y^0 integral gives

$$\int_{-T}^T dy_0 \frac{1}{(y^0)^2 - R^2 + i\epsilon} \simeq \int_{-\infty}^\infty dy_0 \frac{1}{(y^0 - R + i\epsilon)(y^0 + R - i\epsilon)}$$
$$= 2\pi i \left(-\frac{1}{2R}\right), \tag{92}$$

using Cauchy's theorem. Finally one finds

$$\log \mathcal{W}_g = -\frac{i\kappa^2 T m_1 m_2}{16\pi R}. \tag{93}$$

Substituting this result into eq. (84) and using the definition of κ in terms of Newton's constant, $\kappa = \sqrt{16\pi G_N}$, gives

$$V(R) = \frac{G_N m_1 m_2}{R}. \tag{94}$$

The force between the two charges is thus

$$F = -\frac{G_N m_1 m_2}{R^2}, \tag{95}$$

which is Newton's law of gravity.

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[18]In fact, the one-loop diagram enters the exponent to all orders, due to the gravitational analogue of Abelian exponentiation [49–52].

Note that this calculation is closely related to an analysis carried out in [68], which studies non-perturbative quantum corrections to the Newtonian potential. To this aim, the authors define a Wilson line operator obtained by integrating over the worldline of a particle of mass m:

$$W_{\text{wl}} = \exp\left[-im\int_C dt\sqrt{-g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}}\right],\tag{96}$$

where C is the worldline contour, and t the proper time along this path. This operator is equivalent, in the weak field limit, to the operator of eq. (22), as we now show. The weak field expansion of eq. (15) allows us to rewrite eq. (96) as

$$W_{\text{wl}} = \exp\left[-i\int_C dt\sqrt{-\left(\eta_{\mu\nu}+\kappa h_{\mu\nu}\right)p^\mu p^\nu}\right],\tag{97}$$

where we have also taken the mass factor inside the square root and used $p^\mu = mdx^\mu/dt$. Expanding the square root to first order in the graviton field gives

$$W_{\text{wl}} = \exp\left[-im\int_C dt + i\frac{\kappa}{2}\int_C ds\,h_{\mu\nu}p^\mu p^\nu\right],\quad s=\frac{t}{m}.\tag{98}$$

The first term in the exponent is absorbed into the normalisation of the Wilson line operator. The second term is precisely that of eq. (22). That this must be the case follows from the fact that in the eikonal approximation, a particle emitting soft gravitons does not recoil, and thus follows its classical trajectory. Its action must then be given by its classical action, which is indeed the integral over the worldline as in eq. (96). This then fixes the form of the interaction between the eikonal particle and the (soft) graviton field.

B   Normalisation of the solution to Newton's equation

In section 3, we construct eq. (42) for the Newtonian potential Φ, and show that the general solution of the homogeneous equation is given by eq. (45). In this appendix, we show how the constant A_1 can be related to the strength of the source term occuring on the right-hand side of eq. (42). First, note that rewriting this equation in terms of the covariant d'Alambertian operator gives

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\Phi\right) = \left(\frac{\kappa}{2}\right)^2 m_2 e^{-\tau}\delta^{(3)}(\boldsymbol{x}).\tag{99}$$

Integrating over the complete 4-volume on both sides and using the covariant form of the divergence theorem [69], one finds

$$\int_S dS_\mu\sqrt{-g}\,g^{\mu\nu}\,\partial_\nu\Phi = \left(\frac{\kappa}{2}\right)^2\int d\Omega\sqrt{-g}\,m_2\,e^{-\tau}\,\delta^{(3)}(\boldsymbol{x}),\tag{100}$$

where Ω is the 4-volume, and dS_α the element of 3-surface area. The right-hand side gives

$$\left(\frac{\kappa}{2}\right)^2\int d\Omega\sqrt{-g}\,m_2\,e^{-\tau}\,\delta^{(3)}(\boldsymbol{x}) = \left(\frac{\kappa}{2}\right)^2 m_2\int d\tau\,e^{3\tau},\tag{101}$$

where we have used the fact that $\sqrt{-g}\sim e^{4\tau}$, and cancelled spacelike geometric factors in $\sqrt{-g}$ with similar factors in $\delta^{(3)}(\boldsymbol{x})$, consistent with our definition of the delta function in eq. (36). We
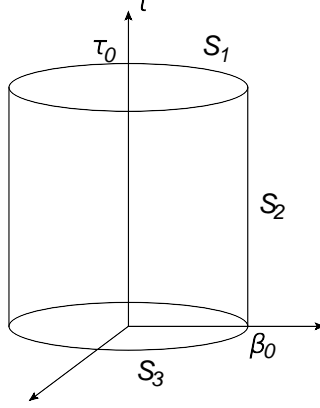
Figure 3: The 3-surface used to normalise the coefficient A_1 in the general solution to the homogeneous Newton equation. The horizontal plane is representative of the spacelike directions, where the radial distance in this plane represents β.

now consider the surface shown in figure 3, and consisting of a cylinder of radius β_0 in the spacelike direction, with upper and lower surfaces at $\tau = \tau_0$ and $\tau = 0$ respectively. Then eq. (101) becomes

$$\left(\frac{\kappa}{2}\right)^2 \int d\Omega \sqrt{-g}\, m_2\, e^{-\tau}\, \delta^{(3)}(\boldsymbol{x}) = \left(\frac{\kappa}{2}\right)^2 m_2 \int_0^{\tau_0} d\tau e^{3\tau}$$

$$= \left(\frac{\kappa}{2}\right)^2 \frac{1}{3} m_2 (e^{3\tau_0} - 1). \tag{102}$$

Splitting up the surface into parts as labelled in figure 3, the surface integrals over S_1, S_2 and S_3 are given by

$$\int_{S_1} dS_\mu \sqrt{-g}\, g^{\mu\nu}\, \partial_\nu \Phi = -4\pi\, e^{3\tau_0} \int_0^{\beta_0} d\beta \sinh^2 \beta \tilde{K}(\beta); \tag{103}$$

$$\int_{S_2} dS_\mu \sqrt{-g}\, g^{\mu\nu}\, \partial_\nu \Phi = \frac{4\pi}{3} \sinh^2 \beta_0 \partial_\beta \left[\tilde{K}(\beta)\right]_{\beta_0} (e^{3\tau_0} - 1); \tag{104}$$

$$\int_{S_3} dS_\mu \sqrt{-g}\, g^{\mu\nu}\, \partial_\nu \Phi = 4\pi \int_0^{\beta_0} d\beta \sinh^2 \beta \tilde{K}(\beta), \tag{105}$$

where we have used eq. (43). Substituting explicitly the general solution of eq. (45) gives a total surface integral of

$$\int_S dS_\mu \sqrt{-g}\, g^{\mu\nu}\, \partial_\nu \Phi = \frac{4\pi}{3} A_1 (e^{3\tau_0} - 1), \tag{106}$$

such that equating this with eq. (102) gives

$$A_1 = \frac{\kappa^2}{16\pi} m_2, \tag{107}$$

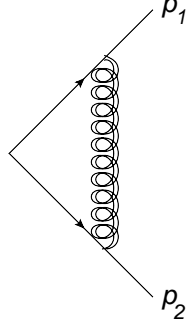as has been used in eq. (46).

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Figure 4: Diagram entering the calculation of the cusp anomalous dimension.

C Field theory calculation of the gravitational cusp anomalous dimension

In this appendix, we detail the calculation of the gravitational cusp anomalous dimension in Minkowski space, using a conventional field theory calculation. First, we consider the diagram shown in figure 4, consisting of a graviton exchange between the two contours on either side of the cusp. We consider the case that p_1 and p_2 are both outgoing, and correspond to different masses m_1 and m_2. Thus, figure 4 is not a cusp in the traditional sense, but rather can be generally embedded into a graph where multiple contours intersect. Using the position space propagator of eq. (88), this gives a contribution

$$\mathcal{F} = \left(-\frac{\kappa}{2}\right)^2 \frac{1}{4\pi^2} p_1^\mu \, p_1^\nu \, P_{\mu\nu,\alpha\beta} \, p_2^\alpha \, p_2^\beta \int_0^\infty ds \int_0^\infty dt \, \frac{1}{(sp_1 - tp_2)^2}. \tag{108}$$

where the tensor $P_{\mu\nu,\alpha\beta}$ is defined in eq. (19). In the $(-,+,+,+)$ metric we are using for gravitational calculations, the cusp angle is given by

$$\cosh \beta_{12} = -\frac{p_1 \cdot p_2}{m_1 m_2}, \tag{109}$$

where $p_i^2 = -m_i^2$. Then eq. (108) can be written, after transforming $s \to s/m_1$, $t \to t/m_2$,

$$\mathcal{F} = -\left(-\frac{\kappa}{2}\right)^2 \frac{1}{4\pi^2} \frac{p_1^\mu \, p_1^\nu \, P_{\mu\nu,\alpha\beta} \, p_2^\alpha \, p_2^\beta}{m_1 m_2} \int_0^\infty ds \int_0^\infty dt \, \frac{1}{s^2 + t^2 - 2st \cosh \beta_{12}}. \tag{110}$$

The integral over s and t is common to the QED case. However, we evaluate this here for completeness. First one sets $s = \lambda t$ to give

$$\int_0^\infty ds \int_0^\infty dt \, \frac{1}{s^2 + t^2 - 2st \cosh \beta_{12}} = \int_0^\infty \frac{dt}{t} \int_0^\infty d\lambda \frac{1}{1 + \lambda^2 - 2 \cosh \beta_{12}}. \tag{111}$$

The t integral contains an ultraviolet and infrared divergence, where the coefficient of the former gives the contribution to the cusp anomalous dimension at one loop order. At a given loop order, care must be taken to consistently separate the IR and UV singular parts, which becomes especially cumbersome when collinear singularities are present. At one loop, however, we may simply evaluate

the t integral by imposing UV and IR cutoffs (a similar procedure is used in the QED analysis of [48]). That is, we may write

$$\int_0^\infty \frac{dt}{t} \to \int_{\Lambda_{\mathrm{UV}}}^{\Lambda_{\mathrm{IR}}} \frac{dt}{t} = \log\left(\frac{\Lambda_{\mathrm{IR}}}{\Lambda_{\mathrm{UV}}}\right). \tag{112}$$

Then one finds

$$\int_0^\infty ds \int_0^\infty dt \, \frac{1}{s^2 + t^2 - 2st\cosh\beta_{12}} = -\log\left(\frac{\Lambda_{\mathrm{UV}}}{\Lambda_{\mathrm{IR}}}\right) \int_{-\cosh\beta_{12}}^\infty d\lambda \frac{1}{\lambda^2 - \sinh^2\beta_{12}}, \tag{113}$$

where we have also completed the square in the λ integral, and transformed $\lambda \to \lambda - \cosh\beta_{12}$. Substituting $\lambda = \sinh\beta_{12}\coth u$, the λ integral can be carried out to give[19]

$$\int_{-\cosh\beta_{12}}^\infty d\lambda \frac{1}{\lambda^2 - \sinh^2\beta_{12}} = \frac{i\pi - \beta_{12}}{\sinh\beta_{12}}. \tag{114}$$

Substituting eqs. (113, 114) into eq. (110) and taking the coefficient of $\log\Lambda_{\mathrm{UV}}$, one finds a contribution to the cusp anomalous dimension given by

$$\Gamma = \frac{\kappa^2}{16\pi^2} \frac{p_1^\mu p_1^\nu P_{\mu\nu,\alpha\beta} p_2^\alpha p_2^\beta}{m_1 m_2} \frac{i\pi - \beta_{12}}{\sinh\beta_{12}}. \tag{115}$$

The kinematic factor is

$$\begin{aligned}
\frac{p_1^\nu P_{\mu\nu,\alpha\beta_{12}} p_2^\alpha p_2^\beta}{m_1 m_2} &= m_1 m_2 \left[\frac{2(p_1 \cdot p_2)^2 - m_1^2 m_2^2}{m_1^2 m_2^2}\right] \\
&= m_1 m_2 \left[2\cosh^2\beta_{12} - 1\right] \\
&= m_1 m_2 \left[1 + 2\sinh^2\beta_{12}\right].
\end{aligned} \tag{116}$$

Finally, one has

$$\Gamma = \frac{\kappa^2}{16\pi^2} m_1 m_2 (i\pi - \beta_{12}) \left[\frac{1}{\sinh\beta_{12}} + 2\sinh\beta_{12}\right]. \tag{117}$$

This is not the whole story. One must also add self-energy diagrams associated with each external line. Rather than calculate these directly, one can surmise their contribution as follows. Firstly, the effect of each self-energy diagram can only depend upon the quantum numbers of a single parton leg, so that the sum over all self-energy contributions has the form

$$\sum_{i=1}^L C p_i^2 = \sum_{i=1}^L C \left[\left(\sum_{i=1}^L p_i\right)^2 - 2\sum_{j>i} p_i \cdot p_j\right], \tag{118}$$

where C is a constant independent of the parton index i, and we have rewritten the momentum dependence on the right-hand side. Using momentum conservation (eq. (69)), one may rewrite eq. (118) as

$$\sum_{i=1}^L C p_i^2 = -2C \sum_i \sum_{j>i} p_i \cdot p_j, \tag{119}$$

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[19]Care is needed with the minus sign in this equation, where the $i\pi$ results from correctly implementing the $i\epsilon$ prescription in the graviton propagator. Alternatively, one may carry out the calculation for one momentum incoming and one outgoing, before analytically continuing $\beta_{12} = i\pi - \beta_{12}$.

and one sees that each pair of external lines is associated with the contribution

$$-2C p_i \cdot p_j = 2C \, m_i \, m_j \left(-\frac{p_i \cdot p_j}{m_i \, m_j} \right)$$
$$= 2C \, m_i \, m_j \, \cosh \beta_{ij}, \tag{120}$$

where we have used the cusp angle definition of eq. (62). Adding this to eq. (117), one may fix the constant C by requiring that the cusp anomalous dimension vanishes at $\beta_{12} = i\pi$, corresponding to a straight line Wilson contour with no cusp. The complete result is then

$$\Gamma = \frac{\kappa^2}{16\pi^2} m_1 \, m_2 \left[(i\pi - \beta_{12}) \left(\frac{1}{\sinh \beta_{12}} + 2 \sinh \beta_{12} \right) + \cosh(\beta_{12}) \right]. \tag{121}$$

Note that in the large β limit, this has the behaviour

$$\Gamma(\beta) \sim \beta e^{\beta}, \tag{122}$$

as discussed in section 3.

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